Indian Statistical Institute

Final Examination 2016-2017

B.Math Third Year

Complex Analysis

Time : 3 Hours Date : 15.11.2016 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $\mathbb{D} := B_1(0)$.

Q1. (10 marks) Let $\mathbb{H}$ be the upper half plane of $\mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{H}$ be a holomorphic function. Prove that f is a constant.

Q2. (10 marks) Identify all the singularities of the following function and determine the nature of each singularity.

$$\frac{z}{e^z - 1}.$$

Q3. (10 marks) Calculate the residues of the following functions at each of the poles:

$$(i)\ \frac{\sin z}{z^2}, \qquad (ii)\ \frac{\cos z}{z^2}.$$

Q4. (10 marks) Let $z = a$ be a pole of order n of a function f. Prove that $z = a$ is a pole of order $n + 1$ of f'.

Q5. (15 marks) Use the residue theorem to compute the following integral

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx.$$

Q6. (15 marks) Let Ω be a simply connected domain and $0 \notin \Omega$. Find all the branches of $z^{\frac{1}{2}}$ in Ω.

Q7. (15 marks) Prove that every biholomorphic map of $\mathbb{C}$ has the form $f(z) = az + b$, where $a(\neq 0)$ and b are in $\mathbb{C}$.

Q8. (15 marks) Let $\epsilon > 0$ and $f : B_{1+\epsilon}(0) \to \mathbb{C}$ be a non-constant holomorphic function. Assume that $|f(z)| = 1$ if $|z| = 1$.

(i) Prove that f has a zero in $\mathbb{D}$.

(ii) Prove that $f(\mathbb{D})$ contains $\mathbb{D}$.

Q9. (15 marks) Let f be a holomorphic function from $\mathbb{D}$ to itself that is not the identity map z. Prove that f has at most one fixed point in $\mathbb{D}$.